JOURNAL OF APPROXIMATION THEORY 14, 188–190 (1975)

The Borsuk–Tietze Linear Extension with a Side Condition

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Communicated by John R. Rice

If D is a closed subset of a compact metric space X and f a continuous function on D then there exists a continuous function F on X with $F = f$ on D. This of course is just a special case of the Tietze extension theorem. If D is a retract of X then there is a linear extension operator T on $C(D)$ to $C(X)$ with $T1 \geqslant 1$, $Tf \geqslant 0$ whenever $f \geqslant 0$ and $Tf = f$ on D. It is interesting to consider the case of D the unit circumference of the unit disk X so that no retract exists. A linear extension operator T can still be defined by setting $Ff(x) = (f, \mu(x))$, where $\mu(x)$ is the Poisson representing measure for the point x in $X \backslash D$. For x in D we interpret $\mu(x)$ to be $\xi(x)$, the unit point mass at x. The existence of a linear extension operator which acts on the bounded continuous functions under hypothesis that X be separable metric is the content of a theorem of Borsuk [1]. (This result has been generalized by Dugundji who removed the separability hypothesis in [2].) Returning to the example of the disk once more the Poisson representing measures ($\mu(x)$ for $x \notin D$) are all absolutely continuous with respect to Lebesgue measure on D. In this note we show that a linear extension operator can be defined for D closed in X a compact metric space so that "representing measures" are absolutely continuous with respect to any pre-assigned probability with support equal to D. Incidentally the argument gives a very geometric proof of both the Tietze Theorem and the Borsuk extension Theorem in this setting.

THEOREM. *Let D be closed in the compact metric space X. Let λ be any Radon probability with closed support equal to D. Then there is a linear extension $T : C(D) \to C(X)$ with the property that for each x in $X \backslash D$ the Radon probability $T^*\xi(x)$ is absolutely λ-continuous.*

Let $K(X)$ denote the Radon probabilities on X and $K(D)$, those with support in D. Then $K(D)$ is easily seen to be a ω^*-compact convex subset of $K(X)$.

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LEMMA. *Let A be a compact convex set in $K(X)$. Then there is a ω^* continuous retract of $K(X)$ onto A.*

Proof. Let $\{f(k): k \in Z_+\}$ be a sequence of functions dense in the boundary of the unit ball of $C(X)$. Then Q given by

$$[Q(\mu, \sigma)]^2 = \sum 2^{-k}(f(k), \mu - \sigma)^2$$

is a metric for the ω^*-topology restricted to the bounded set $K(X)$. Moreover the spherical neighborhoods of the Q metric are strictly convex. Then for each μ in $K(X)$ there is a unique best approximation $\tau(\mu)$ to μ from A. (For uniqueness of best approximations holds whenever we have strict convexity and existence holds whenever we have compactness.) To show continuity of the map $\mu \to \tau(\mu)$ we assume $\mu(n) \to \mu$ and by passing to a subsequence if necessary we can assume $\tau(\mu(n))$ is convergent. If $\tau(\mu(n))$ does not converge to $\tau(\mu)$ we have $\tau(\mu(n))$ disjoint from a Q-ball B about $\tau(\mu)$. Now dist $(\mu, A\backslash B)$ is achieved by compactness; moreover by uniqueness of best approximations the value is strictly less than dist (μ, A). But since dist $(\mu(n), \mu) \to 0$, for sufficiently large n we have $\tau(\mu)$ is a better approximation to $\mu(n)$ than $\tau(\mu(n))$. This contradiction proves the lemma.

We note that retract defined above can be restricted to point masses of X and thus defines a linear extension. Hence the Tietze theorem and the Borsuk theorem in this special case are corollaries of the lemma.

The collection of Radon probabilities which are absolutely λ-continuous does not form a compact set so we must modify the construction to satisfy the side condition. It will be convenient to have a metric bounded by 1 so we replace Q by Q/R where $R = Q\text{-dia}(K(X))$. For each $r \geqslant 1$ we set $A(r) = \{fd\lambda : 0 \leqslant f \leqslant r[\text{mod } \lambda]\} \cap K(X)$. Each $A(r)$ is compact convex for if $\mu(n) = f(n)\, d\lambda$ then $f(n)$ is a sequence in the norm ball of radius r in the space $L_\infty(\lambda)$. Thus a subnet of $f(n)$ converges in the ω^* topology of $L_\infty(\lambda)$ to say f also in $L_\infty(\lambda)$ with $0 \leqslant f \leqslant r$ [mod λ]. The ω^* convergence of $f(n)$ as elements of $L_\infty(\lambda)$ implies ω^* convergence of $f(n)\, d\lambda$ as elements of $C(X)^*$ to $fd\lambda$. Now for any μ in $K(X)$ let $d = Q(\mu, K(D))$ and for $d > 0$ let $\tau(\mu)$ be the unique point of best approximation to μ in $A(r)$ where $r = 1/d$. If $d = 0$ we set $\tau(\mu) = \mu$. Then we need only verify the continuity of the map $\xi(x) \to \tau(\xi(x))$. To this end suppose $\mu(n) \to \mu$. Let $Q(\mu, K(D)) = d > 0$. By compactness we may assume $\tau(\mu(n))$ converges to say σ. Clearly σ is in $A(r)$ where $r = 1/d$. Now if $\sigma \neq \tau(\mu)$ we can place a Q-ball about $\tau(\mu)$ which does not contain σ and argue as in the lemma. Thus $\tau(\mu)$ is the only cluster point of $\tau(\mu(n))$. Finally we verify continuity at the boundary so we now assume dist $(\mu, K(D)) = 0$. Here we only consider the case that μ is a point mass. We claim that $\bigcup A(r)$ is dense in $K(D)$. The closure of $\bigcup A(r)$ is compact convex so it suffices to show each point mass $\xi(x)$ with x in D is in the closure.

Since the support of λ is D there is a sequence of open sets W_n with $\bigcap W_n = \{x\}$ and $\lambda(W_n) > 0$. It follows that there is a sequence of nonnegative continuous functions $h(n)$ with $\int h(n)\, d\lambda = 1$ and support $h(n)$ contained in W_n. The Radon probabilities $\lambda(n) = h(n)\, d\lambda$ can only cluster to a Radon probability with support $\{x\}$. Thus $\bigcup A(r)$ is dense in $K(D)$. Now if $x(n) \to x$, a point in D, then there is a sequence $h(n)\, d\lambda \to \xi(x)$. We also have $\xi(x(n)) \to \xi(x)$. Now if the best approximations do not converge to $\xi(x)$ we obtain a contradiction from the twice used trick of putting a Q-ball about $\xi(x)$.

Now the desired linear extension operator is defined by

$$Tf(x) = (f, \tau(\xi(x)))$$

for x in $X \backslash D$ and $Tf(x) = f(x)$ for x in D.

References

1. K. Borsuk, Über Isomorphie Funktionalraume, *Bull. Internat. Acad. Polon. Sci. Ser. A*, No. 113 (1933), 1–10.
2. J. Dugundji, An extension of Tietze's theorem, *Pacific J. Math.* **1** (1951), 353–367.